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On the cohomology of the holomorph of a finite cyclic group

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Abstract

We determine the mod 2 cohomology algebra of the holomorph of any finite cyclic group whose order is a power of 2.

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1. Outline

The *holomorph* of a group is the semi-direct product of the group with its automorphism group, with respect to the obvious action. The automorphism group of a non-trivial finite cyclic group of order r is well known to be cyclic if and only if the number r is of the kind $r = 4$, $r = p^\rho$, $r = 2p^\rho$ where p is an odd prime; in these cases, the holomorph is thus a split metacyclic group. The mod p cohomology algebra of an arbitrary metacyclic group has been determined in [2]. In this note we will determine the mod 2 cohomology of the holomorph of a cyclic group whose order is a power of 2. Since for odd p the p -primary part of the automorphism group of any cyclic group is cyclic, we thus get a complete description of the mod p cohomology of the holomorph of any (finite) cyclic group. I am indebted to Fred Cohen for having asked whether the cohomology of the holomorph of a cyclic group whose order is a power of 2, this holomorph not being metacyclic when

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the order of the cyclic group is at least 8, may be determined by means of the methods I developed in the 80s.

Let $\rho \geq 2$ and let N be the cyclic group of order $r = 2^\rho$. As usual, we will identify the automorphism group of N with the group $(\mathbb{Z}/2^\rho)^*$ of units of $\mathbb{Z}/2^\rho$, the latter being viewed as a commutative ring. Consider the holomorph

$$G = (\mathbb{Z}/2^\rho) \rtimes (\mathbb{Z}/2^\rho)^* \quad (1.1)$$

of N . For $\rho = 2$, this group comes down to the dihedral group, the group $(\mathbb{Z}/4)^*$ being cyclic of order 2, generated by the class of -1 . Henceforth we suppose that $\rho \geq 3$. Now the group $(\mathbb{Z}/2^\rho)^*$ decomposes as a direct product of a copy of $\mathbb{Z}/2$, generated by the class of -1 , and a copy of $\mathbb{Z}/2^{\rho-2}$, generated by the class of 5. Write $s = 2^{\rho-2}$. The cyclic groups being written multiplicatively, the semi-direct product (1.1) has thus the presentation

$$\langle x, y, z; y^r = 1, x^s = 1, xyx^{-1} = y^5, zyz^{-1} = y^{-1}, [x, z] = 1, z^2 = 1 \rangle, \quad (1.2)$$

the normal cyclic subgroup N being generated by y .

Denote by K_x and K_z the cyclic subgroups of order s and 2 generated by x and z , respectively. The mod 2 cohomology algebra $H^*(N, \mathbb{Z}/2)$ is well known to be generated by a class $\omega_y \in H^1(N, \mathbb{Z}/2)$ and a class $c_y \in H^2(N, \mathbb{Z}/2)$, subject to the relation $\omega_y^2 = (r/2)c_y$. Likewise, the mod 2 cohomology algebra $H^*(K_z, \mathbb{Z}/2)$ is freely generated by a class $\omega_z \in H^1(K_z, \mathbb{Z}/2)$, and the mod 2 cohomology algebra $H^*(K_x, \mathbb{Z}/2)$ is generated by certain classes $\omega_x \in H^1(K_x, \mathbb{Z}/2)$ and $c_x \in H^2(K_x, \mathbb{Z}/2)$, subject to the relation $\omega_x^2 = (s/2)c_x$.

While the cohomology of G may be determined by means of the obvious split extension of N by $K_x \times K_z$, a more economical approach to solving the resulting multiplicative extension problem involves the group \widehat{G} which is given by the presentation

$$\langle x, y, z; y^r = 1, xyx^{-1} = y^5, zyz^{-1} = y^{-1}, [x, z] = 1, z^2 = 1 \rangle \quad (1.3)$$

and projects onto G in an obvious fashion; here the term “more economical” will be justified in Remark 3.7 below. Let ω_1 be the homomorphism from G to $\mathbb{Z}/2$ which sends y to the generator of $\mathbb{Z}/2$ and is trivial on the two other generators x and z of G . Likewise, abusing the notation ω_x and ω_z somewhat, let ω_x and ω_z be the homomorphisms from G to $\mathbb{Z}/2$ which send x and z , respectively, to the generator of $\mathbb{Z}/2$ and are trivial on the respective two other generators. These homomorphisms are well defined and plainly factor through the homomorphisms on K_x and K_z which yield the cohomology classes of these groups denoted by the same symbols. Abusing this notation further, we will denote the composites of ω_1 , ω_x , and ω_z with the projection from \widehat{G} to G by the same symbols as well. Consider the group \widetilde{G} given by the presentation

$$\langle x, \tilde{y}, z; \tilde{y}^{2r} = 1, x\tilde{y}x^{-1} = \tilde{y}^5, z\tilde{y}z^{-1} = \tilde{y}^{-1}, [x, z] = 1, z^2 = 1 \rangle.$$

The obvious homomorphism from \tilde{G} to \hat{G} which sends \tilde{y} to y and the other generators to the generators denoted by the same symbols yields a central extension

$$0 \rightarrow \mathbb{Z}/2 \rightarrow \tilde{G} \rightarrow \hat{G} \rightarrow 1$$

the class of which we denote by $c_2 \in H^2(\hat{G}, \mathbb{Z}/2)$. Since the extension, restricted to $N = \mathbb{Z}/r$, yields the extension representing the class $c_y \in H^2(N, \mathbb{Z}/2)$, the class c_2 restricts to $c_y \in H^2(N, \mathbb{Z}/2)$; likewise, ω_1 restricts to $\omega_y \in H^1(N, \mathbb{Z}/2)$.

Proposition 1.4. *As a graded commutative algebra, $H^*(\hat{G}, \mathbb{Z}/2)$ is generated by $\omega_x, \omega_z, \omega_1, c_2$, subject to the relations*

$$\omega_x^2 = 0, \quad (1.4.1)$$

$$\omega_1^2 = \omega_z \omega_1. \quad (1.4.2)$$

We note that, in characteristic 2, there is no difference between graded commutative and commutative. Here is our main result.

Theorem 1.5. *The mod 2 cohomology of the group G has classes*

$$\omega_3 \in H^3(G, \mathbb{Z}/2), \quad c_4 \in H^4(G, \mathbb{Z}/2)$$

which, under inflation, go to the classes

$$(\omega_1 + \omega_z)c_2 \in H^3(\hat{G}, \mathbb{Z}/2), \quad c_2^2 \in H^4(\hat{G}, \mathbb{Z}/2)$$

such that the mod 2 cohomology algebra of G is generated by

$$c_x, \quad \omega_x, \quad \omega_z, \quad \omega_1, \quad \omega_3, \quad c_4,$$

subject to the relations

$$\omega_1 \omega_3 = 0, \quad (1.5.1)$$

$$c_x \omega_1 = 0, \quad (1.5.2)$$

$$\omega_x^2 = \frac{s}{2} c_x, \quad (1.5.3)$$

$$\omega_1^2 = \omega_z \omega_1, \quad (1.5.4)$$

$$\omega_3^2 = \omega_z \omega_1 c_4 + \omega_z^2 c_4. \quad (1.5.5)$$

Requiring that the classes ω_3 and c_4 restrict to zero in the cohomology of the abelian subgroup of G generated by x and z determines these classes uniquely.

In Section 3 of [2], by means of *homological perturbation theory*, we constructed a free resolution for an arbitrary metacyclic group from a presentation thereof; see [2] and [3] for comments about homological perturbation theory and for more references. The construction may be extended to that of a free resolution for the group G from the presentation (1.2), and an explicit description of the cohomology as well as of various spectral sequences may be derived from it. In this paper we give an alternate somewhat simpler approach which essentially reduces the requisite structural insight to insight into the resolutions for various metacyclic groups which we constructed in [2]. In particular, we shall detect the behaviour of various differentials in certain spectral sequences by inflation to spectral sequences of related group extensions and we shall determine the multiplicative relations in a similar fashion.

2. The additive structure

2.1. The subgroup G_x of G generated by x and y is metacyclic with presentation

$$\langle x, y; y^r = 1, x^s = 1, xyx^{-1} = y^5 \rangle \quad (2.1.1)$$

and fits into the split extension

$$e_x: 1 \rightarrow N \rightarrow G_x \rightarrow K_x \rightarrow 1. \quad (2.1.2)$$

The number

$$\frac{5^{2^{\rho-2}} - 1}{2^\rho} = \frac{(5^{2^{\rho-3}})^2 - 1}{2^\rho} = \frac{5^{2^{\rho-3}} - 1}{2^{\rho-1}} \frac{5^{2^{\rho-3}} + 1}{2}$$

is odd, and we are in the situation of Theorem C of [2] (with the notation of that theorem: for the case $t = 5$ and $p = 2$). In particular, the mod 2 cohomology spectral sequence $(E_r^{s,t}(e_x), d_r)$ for the extension e_x has d_1 equal to zero and, as a commutative algebra, $E_2(e_x)$ is generated by the classes $\omega_x, c_x, \omega_y, c_y$, subject to the relations

$$\omega_x^2 = \frac{s}{2}c_x, \quad \omega_y^2 = \frac{r}{2}c_y.$$

In view of Lemma 4.2.1 of [2], in this spectral sequence, for $j \geq 1$,

$$d_2(c_y^j) = jc_x c_y^{j-1} \omega_y, \quad d_2(c_y^j \omega_y) = 0 \quad (2.1.3)$$

and, as explained on pp. 84 ff. of [2], the spectral sequence collapses from $E_3(e_x)$. For later reference we recall the following simplified version of Theorem C of [2].

Proposition 2.1.4. *The cohomology spectral sequence of the group extension e_x collapses from $E_3(e_x)$, and $H^*(G_x, \mathbb{Z}/2)$ has classes $\omega_3 \in H^3(G_x, \mathbb{Z}/2)$ and $c_4 \in H^4(G_x, \mathbb{Z}/2)$ which restrict to the classes $\omega_y c_y$ and c_y^2 , respectively, in $H^*(N, \mathbb{Z}/2)$ so that, as a*

commutative algebra, $H^*(G_x, \mathbb{Z}/2)$ is generated by $c_x, \omega_x, \omega_1, \omega_3, c_4$, subject to the following relations:

$$\omega_1 \omega_3 = 0, \quad (2.1.4.1)$$

$$c_x \omega_1 = 0, \quad (2.1.4.2)$$

$$\omega_x^2 = \frac{s}{2} c_x, \quad (2.1.4.3)$$

$$\omega_1^2 = 0, \quad (2.1.4.4)$$

$$\omega_3^2 = 0. \quad (2.1.4.5)$$

2.1.5. Later we shall have to resolve various ambiguities related with the multiplicative structure of the group G which we are really interested in. We now describe the corresponding ambiguities for the group G_x . We suppose that a choice of the classes $\omega_3 \in H^3(G_x, \mathbb{Z}/2)$ and $c_4 \in H^4(G_x, \mathbb{Z}/2)$ has been made. In dimensions 1, 2, 3, 4 the following monomials then constitute bases:

$$H^1(G_x, \mathbb{Z}/2): \quad \omega_1, \omega_x,$$

$$H^2(G_x, \mathbb{Z}/2): \quad \omega_1 \omega_x, c_x,$$

$$H^3(G_x, \mathbb{Z}/2): \quad \omega_x c_x, \omega_3,$$

$$H^4(G_x, \mathbb{Z}/2): \quad c_x^2, c_4, \omega_3 \omega_x.$$

Hence requiring that, under the restriction to $H^*(K_x, \mathbb{Z}/2)$, ω_3 and c_4 go to zero determines ω_3 uniquely and c_4 up to the term $\omega_3 \omega_x$.

2.2. The subgroup G_z of G generated by z and y is metacyclic of the form $G_z = N \rtimes K_z$ and thus fits into the split extension

$$e_z: 1 \rightarrow N \rightarrow G_z \rightarrow K_z \rightarrow 1.$$

The (images of the) classes ω_z, ω_1, c_2 in $H^*(G_z, \mathbb{Z}/2)$ generate the constituent $E_2(e_z)$ of the mod 2 cohomology spectral sequence of e_z which necessarily collapses from E_2 whence, as a (graded) commutative algebra, $H^*(G_z, \mathbb{Z}/2)$ is generated by these classes. By Theorem B of [2], the relation (1.4.2) is defining, that is, $H^*(G_z, \mathbb{Z}/2)$ is generated by these classes subject to the relation (1.4.2).

2.3. The projection from \widehat{G} to the cyclic group generated by x yields the group extension

$$\widehat{e}: 1 \rightarrow G_z \rightarrow \widehat{G} \rightarrow \mathbb{Z} \rightarrow 1. \quad (2.3.1)$$

We now give a somewhat more precise version of Proposition 1.4 in the introduction.

Proposition 2.3.2. *The spectral sequence of the extension \hat{e} collapses from E_2 and, as a graded commutative algebra, $H^*(\hat{G}, \mathbb{Z}/2)$ is generated by $\omega_x, \omega_z, \omega_1, c_2$, subject to the relations (1.4.1) and (1.4.2).*

Proof. Since every multiplicative generator of the cohomology of G_z is in the image of the restriction map from \hat{G} to G_z , the multiplicative properties of the spectral sequence of the extension \hat{e} entail that the spectral sequence collapses from E_2 . The relation (1.4.1) holds for degree reasons. Moreover, since in $H^*(G_z, \mathbb{Z}/2)$ the relation (1.4.2) holds, for suitable coefficients $a, b \in \mathbb{Z}/2$,

$$\omega_1^2 = \omega_z \omega_1 + a \omega_z \omega_x + b \omega_1 \omega_x.$$

Restricting this relation to the subgroup of \hat{G} (of the kind $K_z \times \mathbb{Z}$) generated by x and z we see that $a = 0$. Likewise, restricting the relation to the subgroup \hat{G}_x of \hat{G} (of the kind $N \rtimes \mathbb{Z}$) generated by x and y , by virtue of Proposition 2.1.4, since $\omega_1^2 = 0$ in $H^2(\hat{G}_x, \mathbb{Z}/2)$, we conclude that $b = 0$. \square

2.4. The projection from G to the cyclic group generated by x yields the group extension

$$e: 1 \rightarrow G_z \rightarrow G \rightarrow K_x \rightarrow 1 \quad (2.4.1)$$

in an obvious fashion. In view of Section 2.2 above, as a graded $\mathbb{Z}/2$ -algebra, the cohomology $H^*(G_z, \mathbb{Z}/2)$ of G_z is generated by ω_z, ω_1, c_2 , subject to the relation (1.4.2).

Proposition 2.4.2. *The cohomology spectral sequence $(E_r(e), d_r)$ of the group extension e has*

$$d_2(\omega_1) = 0, \quad d_2(c_2) = \omega_1 c_x$$

and collapses from $E_3(e)$. Furthermore, $H^*(G, \mathbb{Z}/2)$ has a class $\omega_3 \in H^3(G, \mathbb{Z}/2)$ which restricts to $(\omega_1 + \omega_z)c_2 \in H^3(G_z, \mathbb{Z}/2)$ and a class $c_4 \in H^4(G, \mathbb{Z}/2)$ which restricts to $c_2^2 \in H^4(G_z, \mathbb{Z}/2)$ so that, as a commutative algebra, $H^*(G, \mathbb{Z}/2)$ is generated by $c_x, \omega_x, \omega_z, \omega_1, \omega_3, c_4$.

Proof. The class ω_1 is manifestly an infinite cycle. For suitable coefficients $\eta, \varepsilon \in \mathbb{Z}/2$, the spectral sequence $(E_r(e), d_r)$ necessarily has

$$d_2(c_2) = \eta \omega_1 c_x + \varepsilon \omega_z c_x.$$

Consider the morphism

$$\begin{array}{ccccccc} 1 & \longrightarrow & N & \longrightarrow & G_x & \longrightarrow & K_x \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \text{Id} \\ 1 & \longrightarrow & G_z & \longrightarrow & G & \longrightarrow & K_x \longrightarrow 1 \end{array} \quad (2.4.3)$$

of extensions from e_x to e . Restricting the spectral sequence $(E_r(e), d_r)$ to the extension e_x we deduce from (2.1.3) that $\eta = 1$. Likewise, restricting the spectral sequence to the extension

$$e^\times : 1 \rightarrow K_z \rightarrow K_z \times K_x \rightarrow K_x \rightarrow 1 \quad (2.4.4)$$

we see that $\varepsilon = 0$. Consequently, in the spectral sequence $(E_r(e), d_r)$,

$$d_2(c_2) = c_x \omega_1 \quad (2.4.5)$$

whence, in view of the multiplicative structure of the spectral sequence, for $j \geq 1$,

$$d_2(c_2^j) = j c_2^{j-1} c_x \omega_1, \quad d_2(c_2^j \omega_1) = j c_2^{j-1} c_x \omega_1^2 = j c_2^{j-1} c_x \omega_1 \omega_z. \quad (2.4.6)$$

In particular, $d_2 c_2^2 = 0$. We assert that c_2^2 is an infinite cycle. Indeed, the value of $d_3 c_2^2$ lies in $E_3^{3,2}(e)$. Restricting the spectral sequence $(E_r(e), d_r)$ to the two extensions e_x and e^\times we deduce that $d_3 c_2^2$ is zero, and we are left with

$$d_4 c_2^2 \in E_4^{4,1}(e).$$

The same kind of detection principle as that used above shows that the only possibility for $d_4 c_2^2$ to be non-zero is that, for a suitable coefficient $a \in \mathbb{Z}/2$,

$$d_4 c_2^2 = a \omega_z^3 \omega_x \omega_y.$$

However, the obvious projection map yields a morphism of extensions from \hat{e} to e and hence a morphism of spectral sequences via the inflation map. By Proposition 2.3.2, the mod 2 cohomology spectral sequence $(E_r^{s,t}(\hat{e}), d_r)$ collapses from E_2 . Hence the coefficient a is zero, that is, $d_4 c_2^2$ is zero. Finally, since the extension e is split, a non-zero differential of the spectral sequence $(E_r^{s,t}(e), d_r)$ cannot hit the baseline. Consequently $d_5 c_2^2$ is zero whence c_2^2 is an infinite cycle.

Likewise $(\omega_1 + \omega_z)c_2$ is an infinite cycle. The multiplicative properties of the spectral sequence $(E_r^{s,t}(e), d_r)$ imply that this spectral sequence collapses from $E_3(e)$. We conclude that, apart from the multiplicative generators $c_x, \omega_x, \omega_z, \omega_1$, the cohomology algebra $H^*(G, \mathbb{Z}/2)$ has an additional generator $\omega_3 \in H^3(G, \mathbb{Z}/2)$ which restricts to $(\omega_1 + \omega_z)c_2 \in H^3(G_z, \mathbb{Z}/2)$ and another generator $c_4 \in H^4(G, \mathbb{Z}/2)$ which restricts to $c_2^2 \in H^4(G_z, \mathbb{Z}/2)$ such that, as an algebra, $H^*(G, \mathbb{Z}/2)$ is generated by these six generators. \square

2.4.7. We suppose that a choice of the classes $\omega_3 \in H^3(G, \mathbb{Z}/2)$ and $c_4 \in H^4(G, \mathbb{Z}/2)$ has been made. In dimensions 3 and 4 the following monomials then constitute bases:

$$H^3(G, \mathbb{Z}/2): \quad \omega_x c_x, \omega_3; \quad \omega_z \omega_1 \omega_x, \omega_z c_x; \quad \omega_z^2 \omega_1, \omega_z^2 \omega_x, \omega_z^3,$$

$$H^4(G, \mathbb{Z}/2): \quad c_x^2, c_4, \omega_3 \omega_x; \quad \omega_z \omega_x c_x, \omega_z \omega_3; \quad \omega_z^2 \omega_1 \omega_x, \omega_z^2 c_x; \quad \omega_z^3 \omega_1, \omega_z^3 \omega_x, \omega_z^4.$$

Requiring that ω_3 and c_4 restrict to zero in the cohomology of the abelian subgroup of G generated by x and z leaves the following ambiguities:

$$\begin{aligned}\omega_3: & \quad \omega_z^2 \omega_1, \quad \omega_z \omega_1 \omega_x, \\ c_4: & \quad \omega_z^3 \omega_1, \quad \omega_z^2 \omega_1 \omega_x, \quad \omega_3 \omega_x.\end{aligned}$$

Requiring that, furthermore, the classes ω_3 and c_4 go, under inflation, to the classes

$$(\omega_1 + \omega_z)c_2 \in H^3(\widehat{G}, \mathbb{Z}/2), \quad c_2^2 \in H^4(\widehat{G}, \mathbb{Z}/2),$$

respectively, as asserted in Theorem 1.5, determines the classes $\omega_3 \in H^3(G, \mathbb{Z}/2)$ and $c_4 \in H^4(G, \mathbb{Z}/2)$ uniquely, since the classes $\omega_z^2 \omega_1$, $\omega_z \omega_1 \omega_x$ and $\omega_z^3 \omega_1$, $\omega_z^2 \omega_1 \omega_x$, $\omega_3 \omega_x$ go to linearly independent classes in $H^*(\widehat{G}, \mathbb{Z}/2)$. Henceforth we suppose that these choices for ω_3 and c_4 which determine them uniquely have been made. Then, under (2.4.3), the generator $\omega_3 \in H^3(G, \mathbb{Z}/2)$ goes to the generator in $H^3(G_x, \mathbb{Z}/2)$ denoted in 2.1.4 by the same symbol and $c_4 \in H^4(G, \mathbb{Z}/2)$ goes to one of the two possible choices for the class in $H^4(G_x, \mathbb{Z}/2)$ denoted c_4 in 2.1.4 and hence resolves the ambiguity left open in Section 2.1.5.

These observations yield, in particular, a proof of Theorem 1.5, as far as the additive structure is concerned.

Remark 2.4.8. The value $d_2(c_2) = \omega_1 c_x$ of the class c_2 under the differential d_2 in the spectral sequence of the extension e may also be determined by means of Theorem 1 in [1].

3. The multiplicative structure

The obvious projection from \widehat{G} to G yields a central extension

$$1 \rightarrow \mathbb{Z} \rightarrow \widehat{G} \rightarrow G \rightarrow 1. \quad (3.1)$$

We realize this central extension geometrically by a fiber bundle of the kind

$$B\widehat{G} \rightarrow BG \rightarrow BS^1 \quad (3.2)$$

where the map from BG to BS^1 represents the cohomology class c_x ; here S^1 is the circle group. By Proposition 2.3.2, as a graded $\mathbb{Z}/2$ -algebra, the cohomology $H^*(\widehat{G}, \mathbb{Z}/2)$ of the fiber is generated by $\omega_x, \omega_z, \omega_1, c_2$, subject to the relations $\omega_1^2 = \omega_z \omega_1$ and $\omega_x^2 = 0$.

Consider the mod 2 spectral sequence $(E_r^{s,t}, d_r)$ for (3.2). It has

$$E_2^{*,t} = \mathbb{Z}/2[c_x] \otimes H^t(\widehat{G}, \mathbb{Z}/2)$$

and, as usual, we will write $E_\infty = E_\infty(H^*(G, \mathbb{Z}/2))$ for the associated graded algebra coming from the filtration in terms of powers of the multiplicative generator c_x . Given $p, q \geq 0$, this filtration has the form

$$H^{2p,q} \subseteq H^{2p-2,q+2} \subseteq \dots \subseteq H^{0,q+2p} = H^{q+2p}(G, \mathbb{Z}/2).$$

Since ω_z and ω_1 are restrictions of cohomology classes of G , in the spectral sequence, they are infinite cycles. Moreover, in view of (2.4.5), the spectral sequence has

$$d_2(c_2) = \omega_1 c_x. \quad (3.3)$$

More precisely, for degree reasons, for suitable coefficients a, b, c in $\mathbb{Z}/2$,

$$d_2(c_2) = a\omega_1 c_x + b\omega_x c_x + c\omega_z c_x.$$

Restricting the extension (3.1) to the extension

$$1 \rightarrow \mathbb{Z} \rightarrow \widehat{G}_x \rightarrow G_x \rightarrow 1$$

we see that $a = 1$ and $b = 0$ and restricting the extension (3.1) to the trivial extension

$$1 \rightarrow 1 \rightarrow G_z \rightarrow G_z \rightarrow 1$$

so that BS^1 is replaced with a point we see that $c = 0$. Likewise

$$d_2(\omega_1 c_2) = \omega_1^2 c_x = \omega_z \omega_1 c_x, \quad d_2(\omega_z c_2) = \omega_z \omega_1 c_x,$$

and $(\omega_1 + \omega_z)c_2$ is an infinite cycle, indeed, arises from the class

$$\omega_3 \in H^3(G, \mathbb{Z}/2) = H^{0,3}$$

under the surjection from $H^{0,3}$ to $E_\infty^{0,3}$. Furthermore, $d_2(c_2^2) = 0$ whence c_2^2 is an infinite cycle. It arises from the class $c_4 \in H^4(G, \mathbb{Z}/2) = H^{0,4}$ under the surjection from $H^{0,4}$ to $E_\infty^{0,4}$. Abusing notation somewhat, we will write

$$\omega_3 = (\omega_1 + \omega_z)c_2, \quad c_4 = c_2^2.$$

Proposition 3.4. *The associated graded algebra $E_\infty(H^*(G, \mathbb{Z}/2))$ is generated by $c_x, \omega_x, \omega_z, \omega_1, \omega_3, c_4$, subject to the relations*

$$\omega_1 \omega_3 = 0, \quad (1.5.1(\infty))$$

$$c_x \omega_1 = 0, \quad (1.5.2(\infty))$$

$$\omega_x^2 = \frac{s}{2} c_x, \quad (1.5.3)$$

$$\omega_1^2 = \omega_z \omega_1, \quad (1.5.4(\infty))$$

$$\omega_3^2 = \omega_z \omega_1 c_4 + \omega_z^2 c_4. \quad (1.5.5(\infty))$$

Proof. Indeed, since $\omega_1^2 = \omega_1 \omega_z \in H^2(\widehat{G}, \mathbb{Z}/2)$, we have

$$\omega_1 \omega_3 = \omega_1 (\omega_1 + \omega_z) c_2 = (\omega_1^2 + \omega_1 \omega_z) c_2 = 0,$$

that is, the relation (1.5.1(∞)) holds. The relation (1.5.2(∞)) is an immediate consequence of (3.3). The relation (1.5.3) holds already in $H^2(K_x, \mathbb{Z}/2)$, and the relation (1.5.4(∞)) holds in $H^2(\widehat{G}, \mathbb{Z}/2)$. Finally,

$$\omega_3^2 = \omega_1^2 c_4 + \omega_z^2 c_4 = \omega_1 \omega_z c_4 + \omega_z^2 c_4,$$

whence the relation (1.5.5(∞)) is satisfied. \square

We now complete the proof of Theorem 1.5 by verifying that the relations (1.5.1(∞)), (1.5.2(∞)), (1.5.4(∞)), and (1.5.5(∞)) hold in $H^*(G, \mathbb{Z}/2)$. This amounts to showing that the multiplicative extension problem with reference to the filtration coming from powers of the generator c_x is trivial. To this end we examine the possible ambiguities of the corresponding relations in $H^*(G, \mathbb{Z}/2)$.

The relation (1.5.4(∞)) arises from a relation of the kind

$$\omega_1^2 = \omega_z \omega_1 + a c_x$$

for some $a \in \mathbb{Z}/2$. Restricting this relation to the abelian subgroup of G generated by x and z we find that $a = 0$ whence the relation (1.5.4) holds.

Likewise the relation (1.5.2(∞)) arises from a relation of the kind

$$c_x \omega_1 = a \omega_z c_x + b \omega_x c_x,$$

for suitable coefficients $a, b \in \mathbb{Z}/2$. When we restrict this relation to the abelian subgroup of G generated by x and z we find $a = b = 0$, that is, the relation (1.5.2) holds.

Since $\omega_1 \omega_3$ restricts to zero in the cohomology of \widehat{G} , the relation (1.5.1(∞)) arises from a relation of the kind

$$\omega_1 \omega_3 = b(\omega_x, \omega_z, \omega_1) c_x + d c_x^2 \quad (1.5.1')$$

where b is a quadratic polynomial in $\omega_x, \omega_z, \omega_1$ and where $d \in \mathbb{Z}/2$. The group $H^2(\widehat{G}, \mathbb{Z}/2)$ has dimension 4, a basis being given by

$$\omega_1 \omega_z, \quad \omega_1 \omega_x, \quad \omega_z^2, \quad \omega_z \omega_x; \quad (3.5)$$

notice that $\omega_x^2 = 0$. The relations $\omega_1^2 = \omega_1 \omega_z$ and $\omega_1 c_x = 0$ in the cohomology of G have already been established. Thus only the monomials $\omega_x \omega_z$ and ω_z^2 may yield non-zero contributions to $b(\omega_x, \omega_z, \omega_1) c_x$, that is, for some $\beta, \gamma \in \mathbb{Z}/2$,

$$\omega_1 \omega_3 = \beta \omega_z \omega_x c_x + \gamma \omega_z^2 c_x + d c_x^2. \quad (1.5.1'')$$

When we restrict the relation (1.5.1'') to the subgroup of G (of the kind $K_z \times K_x$) generated by x and z , since ω_1 restricts to zero, we see that $\beta = \gamma = 0$ and $d = 0$, that is, the relation (1.5.1) holds.

We will now resolve the ambiguities for the relation (1.5.5(∞)). The monomials

$$c_4, \quad \omega_1 \omega_z^2 \omega_x, \quad \omega_z^3 \omega_x, \quad \omega_z^3 \omega_1, \quad \omega_3 \omega_1, \quad \omega_3 \omega_z, \quad \omega_3 \omega_x \quad (3.6)$$

constitute a basis of $H^4(\widehat{G}, \mathbb{Z}/2)$. Since, in the cohomology of G , $\omega_1 c_x = 0$, non-zero contributions to the square ω_3^2 may at first arise from the products of each of the monomials $c_4, \omega_z^3 \omega_x, \omega_3 \omega_z, \omega_3 \omega_x$ with c_x . However, ω_3^2 restricts to zero in the cohomology of the metacyclic subgroup G_x of G generated by x and y , and the monomials $c_4 c_x$ and $\omega_3 \omega_x c_x$ are non-zero in $H^6(G_x, \mathbb{Z}/2)$. Likewise, ω_3^2 restricts to zero in the cohomology of the abelian subgroup $K_x \times K_z$ of G generated by x and z , and $\omega_z^3 \omega_x c_x$ is non-zero in $H^6(K_x \times K_z, \mathbb{Z}/2)$. Hence, among the monomials which are *linear* in c_x , the square ω_3^2 may involve at most the term $\omega_3 \omega_z c_x$.

We now show that monomials which are *quadratic* in c_x cannot yield a non-zero contribution to ω_3^2 . Indeed, these monomials arise from the monomials (3.5) since these constitute a basis of $H^2(\widehat{G}, \mathbb{Z}/2)$. Among the four monomials of this kind, the monomials $\omega_1 \omega_z$ and $\omega_1 \omega_x$ will not contribute anything non-zero since $\omega_1 c_x = 0$. Furthermore, the monomials $\omega_z^2 c_x^2$ and $\omega_z \omega_x c_x^2$, restricted to the abelian subgroup $K_x \times K_z$ of G generated by x and z , yield non-zero elements of $H^6(K_x \times K_z, \mathbb{Z}/2)$. Thus, since the square ω_2^3 restricts to zero in $H^6(K_x \times K_z, \mathbb{Z}/2)$, ω_3^2 cannot involve a monomial which is *quadratic* in c_x .

Summing up, we conclude that, for some $\gamma \in \mathbb{Z}/2$, possibly zero,

$$\omega_3^2 = \omega_z \omega_1 c_4 + \omega_z^2 c_4 + \gamma \omega_z \omega_3 c_x. \quad (1.5.5')$$

The following argument rules out the possibility of a non-zero γ . Consider the subgroup A of G generated by $x, y^{r/2}$, and z . This subgroup decomposes as a direct product $A \cong \mathbb{Z}/2 \times K_z \times K_x$ where the unlabelled copy of $\mathbb{Z}/2$ is generated by $y^{r/2}$. As a graded commutative algebra, the mod 2 cohomology of this group is generated by a class $\omega: A \rightarrow \mathbb{Z}/2$ which is the obvious projection to the unlabelled copy of $\mathbb{Z}/2$, together with the classes ω_z, ω_x , and c_x , subject to the relation $\omega_x^2 = 0$, where the notation is abused. Under the restriction map from $H^*(G, \mathbb{Z}/2)$ to $H^*(A, \mathbb{Z}/2)$, the classes ω_z, ω_x , and c_x go to the classes denoted by the same symbols, the class ω_1 goes to zero, the class ω_3 to $\omega_z \omega^2$, and the class c_4 to ω^4 . Moreover, the relation (1.5.5') passes to the relation

$$\omega_z^2 \omega^4 = \omega_z^2 \omega^4 + \gamma \omega_z^2 \omega^2 c_x$$

whence $\gamma = 0$. Hence the relation (1.5.5) holds in the cohomology of G . This completes the proof of Theorem 1.5.

Remark 3.8. Our method involving the group \widehat{G} or, more precisely, the fiber bundle (3.2), rather than the obvious split extension of N by $K_x \times K_z$, leads to a somewhat more economical approach to solving the multiplicative extension problem since then the only ambiguities for this extension problem come from the powers of the generator c_x . The ambiguities for the multiplicative extension problem arising from the split extension of N by $K_x \times K_z$ come from arbitrary monomials in the generators ω_x , c_x , and ω_z .

Remark 3.9. The group G may as well be written as a split extension $G = G_x \rtimes K_z$ and, in view of Proposition 2.1.4, with reference to the induced filtration of the mod 2 cohomology algebra, the associated graded algebra is just the tensor product of $H^*(G_x, \mathbb{Z}/2)$ and $H^*(K_z, \mathbb{Z}/2)$. The relations (1.5.4) and (1.5.5) show that the multiplicative extension problem is non-trivial, though.

Remark 3.10. The group N has order 2^ρ , and we have made the assumption that $\rho \geq 3$. Theorem 1.5 shows that the mod 2 cohomology algebra of the holomorph $N \rtimes \text{Aut}(N)$ is then independent of ρ . Given ρ , consider the group $N^\sharp = \mathbb{Z}/2^{\rho+1}$. The obvious surjection from N^\sharp to $N = \mathbb{Z}/2^\rho$ induces a surjection from $N^\sharp \rtimes \text{Aut}(N^\sharp)$ to $N \rtimes \text{Aut}(N)$ but, beware, this surjection does *not* induce an isomorphism between the mod 2 cohomology algebras. Indeed, under the induced morphism in cohomology, the multiplicative generators c_x , ω_3 , and c_4 go to zero.

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